Facility location II.

Chapter 10

Location-Allocation Model
Plant Location Model
Network Location Models

Facility location models

- Rectilinear Facility Location Problems

  Location of a new facility in relation to other facilities

  - Single Facility Minisum Location Problem
  - Single Facility Minimax Location Problem
- Network Location Models
  - 1-Median Problem (Minisum)
  - 1-Center Problem (Mimimax)
- Location Allocation Models

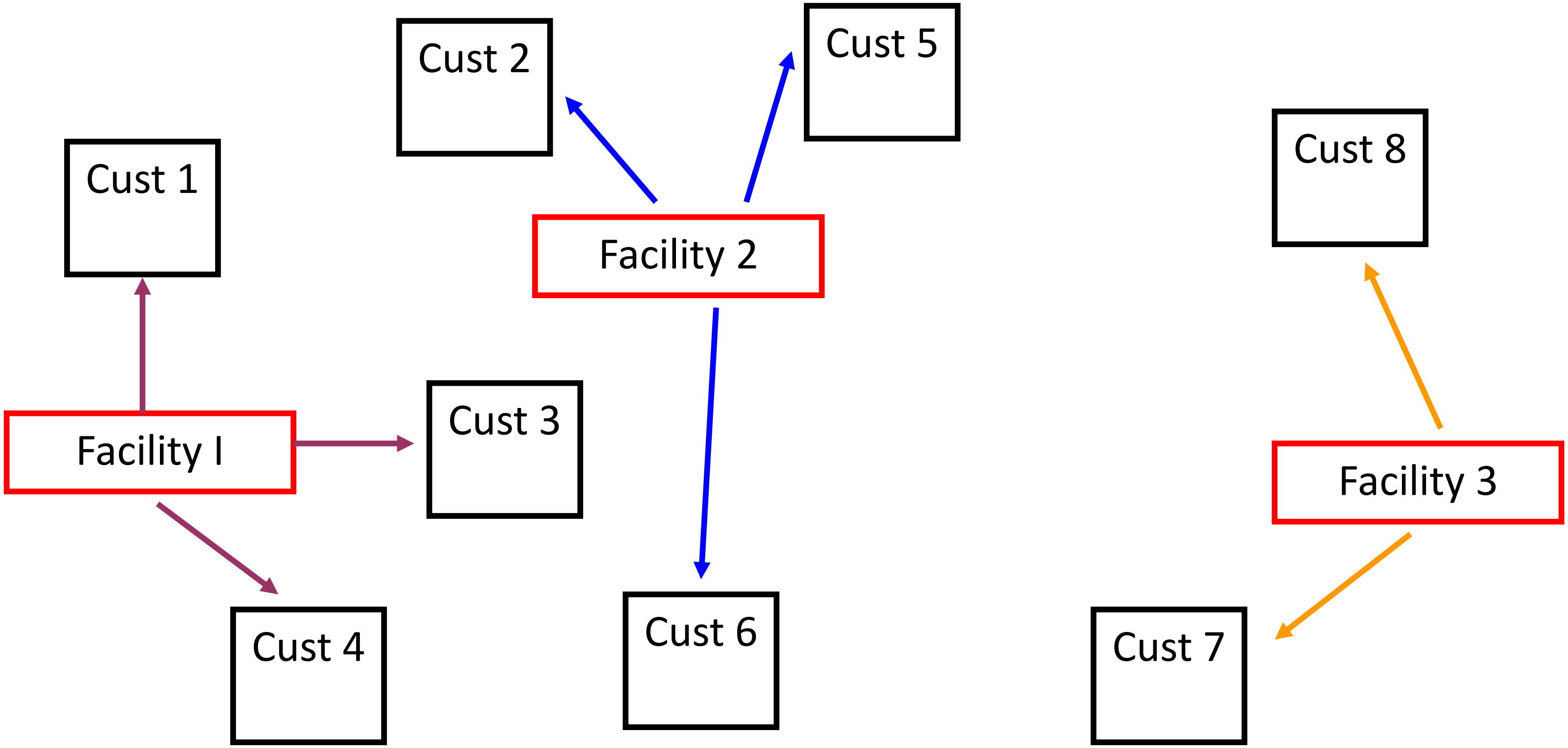
  Determination of the number of new facilities, their location and the customer groups which will be served by each one of them
- Plant Location Problem

  Possible locations of the new facilities are known. Selection of the number of new facilities and customer groups which will be served by each one of them

Location-Allocation Model

- Involves determination of:
  - optimum number of new facilities
  - where new facilities are to be located
  - which customers (of existing facilities) should be served by each new facility
- The objective is to:
  - Minimize the total material movement cost
  - Minimize the total fixed cost

Location-Allocation Model



Location-Allocation Model

$$\min \psi = \sum_{j=1}^{n} \sum_{i=1}^{m} z_{ji} w_{ji} d(X_j, P_k) + g(n)$$

$$s.t.$$

$$\sum_{j=1}^{n} z_{ji} = 1 \quad for \quad i = 1, \ldots, m$$

Ensures that each existing facility interacts with only one new facility

Where

- Ψ is the total cost per unit of time
- *n* is the number of new facilities (*n* = 1, …, *m*)
- w_{ji} is the cost per unit of time per unit distance if new facility *j* interacts with existing facility
- z_{ji} = 1 if the new facility *j* interacts with the existing facility *i*, 0 otherwise
- *g*(*n*) is the cost per unit of time of providing *n* new facilities
- $d(X_j, P_k)$ is the rectilinear distance between existing and new facilities

Location-Allocation Model

Solution

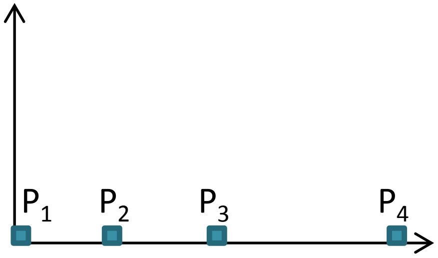
- To solve the model an enumeration procedure is implemented
- For *m* customers and *n* facilities, the number of possible alternatives:

$$S(n,m)=\sum_{k=0}^{n-1}\frac{(-1)^k(n-k)^m}{k!(n-k)}$$

- **Procedure:**
  - Enumerate all of the allocation combinations for each value of *n*
  - Determine the optimum location for each new facility for each allocation combination
  - Specify the minimum cost solution

Location-Allocation Model

Example 1



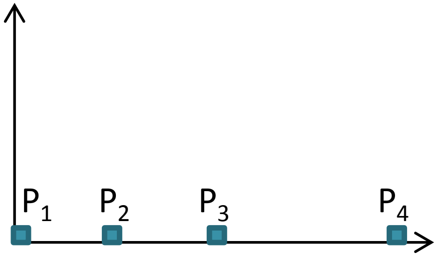
- 4 customers:
    - $P_1(0,0)$, $P_2(3,0)$, $P_3(6,0)$, $P_4(12,0)$
    - $w_1 = w_2 = w_3 = 1$ and $w_4 = 2$
    - The cost of setting n new facilities: $g(n) = 5n$

- How many facilities should be built?
- Which customers should be served by which facilities?
- Where these facilities should be located?
- What is the minimum total cost?

- Solution: There are four different scenarios
    1. n = 1
    2. n = 2
    3. n = 3
    4. n = 4

Location-Allocation Model

Example 1



- When ***n=1,*** then:

  - solve simply by single facility location model:

- Minisum Location Problem: - x coordinate

a_i	w_i	$\sum w_i$
0	1	1
3	1	2
6	1	3
12	2	5

Half the total weight 5/2 = 2.5

Facility location:

X=6 y=0

$$TC = \sum_{j=1}^{n} \sum_{i=1}^{m} z_{ji} w_{ji} d(X_j, P_k) + g(n)$$

Total cost:

TC(1) = 1*(6-0) + 1*(6-3) + 1*(6-6) + 2*(12-6) + 5*1 = <u>26</u>

Location-Allocation Model

Example 1

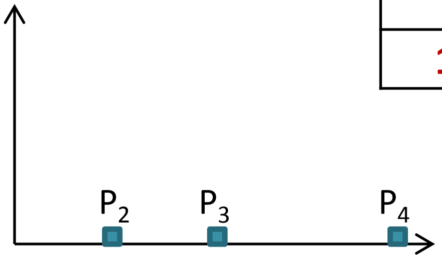
- When ***n* = 2**, then:

	New Facility 1	New Facility 2
a	**P_1**	**P_2, P_3, P_4**
b	P_1, P_2	P_3, P_4
c	P_1, P_2, P_3	P_4
…	…	…

Find locations and total costs for these cases

- Solve each case by Minisum technique:
  - **Case a:**

a_i	w_i	$\sum w_i$
3	1	1
6	1	**2**
12	2	4



- For P_1 => (0,0)

For P_2, P_3, P_4 => all the points between (6,0) and (12,0)

Total cost: TC(2a) for (0,0) and (6,0) = 1*(0-0) + 1*(6-3) + 1*(6-6) + 2*(12-6) + 5*2 = <u>25</u>

TC(2a) for (0,0) and (12,0) = 1*(0-0) + 1*(12-3) + 1*(12-6) + 2*(12-12) + 5*2 = <u>25</u>

Location-Allocation Model

Example 1

- When ***n* = 3**, then:

New Facility 1	New Facility 2	New Facility 3
P_1	P_2	$P_{3,}$ P_4
P_1	P_2, P_3	P_4
P_1, P_2	P_3	P_4
…	…	…

- Find locations and total costs for all of these alternatives.
- Select the case with the lowest cost

- When ***n* = 4**, then:

New Facility 1	New Facility 2	New Facility 3	New Facility 4
P_1	P_2	$P_{3,}$	P_4

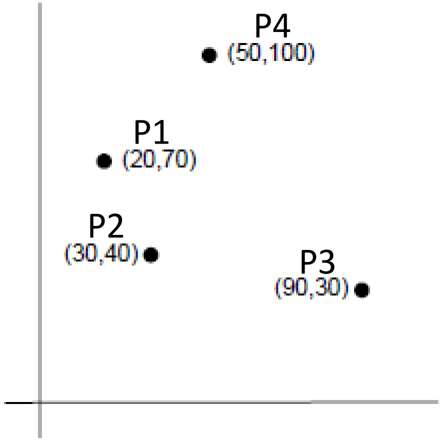
Location-Allocation Model

Example 2

- Coffee shops are planned to be placed in an office building. The office tenants are located at P_1(20,70), P_2(30,40), P_3(90,30) and P_4(50,100). 50 persons per day are expected to visit the first office, 30 the second office, 70 the third office and 60 the last office. 70% of visitors are expected to drop by the coffee shop. Each unit distance which a customer has to travel costs the owner of the coffee shops the loss of \$0.25 in revenue. The daily operating cost of *n* shops is \$5000*n*

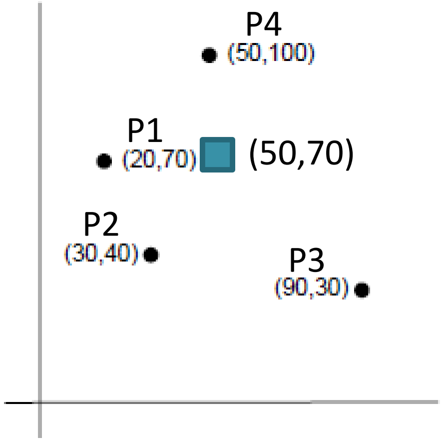
➢ Determine the number of coffee shops and their locations

	No. of People/Day	Coffee Shop Visitors (70%)
P_1	50	35
P_2	30	21
P_3	70	49
P_4	60	42



Location-Allocation Model

Example 2



- *When **n=1***

- Minisum algorithm:

a_i	w_i	Σw_i
20	35	35
30	21	56
50	42	98
90	49	147

x* = 50

b_i	w_i	Σw_i
30	49	49
40	21	70
70	35	105
100	42	147

y* = 70

$(x^*,y^*) = (50,70)$

$TC(1) = 5{,}000(1) + 0.25[35(30) + 21(50) + 49(80) + 42(30)] = 6{,}820$

$$TC = \sum_{j=1}^{n}\sum_{i=1}^{m} z_{ji} w_{ji} d(X_j, P_k) + g(n)$$

Location-Allocation Model

Example 2

- *When* ***n>=2***

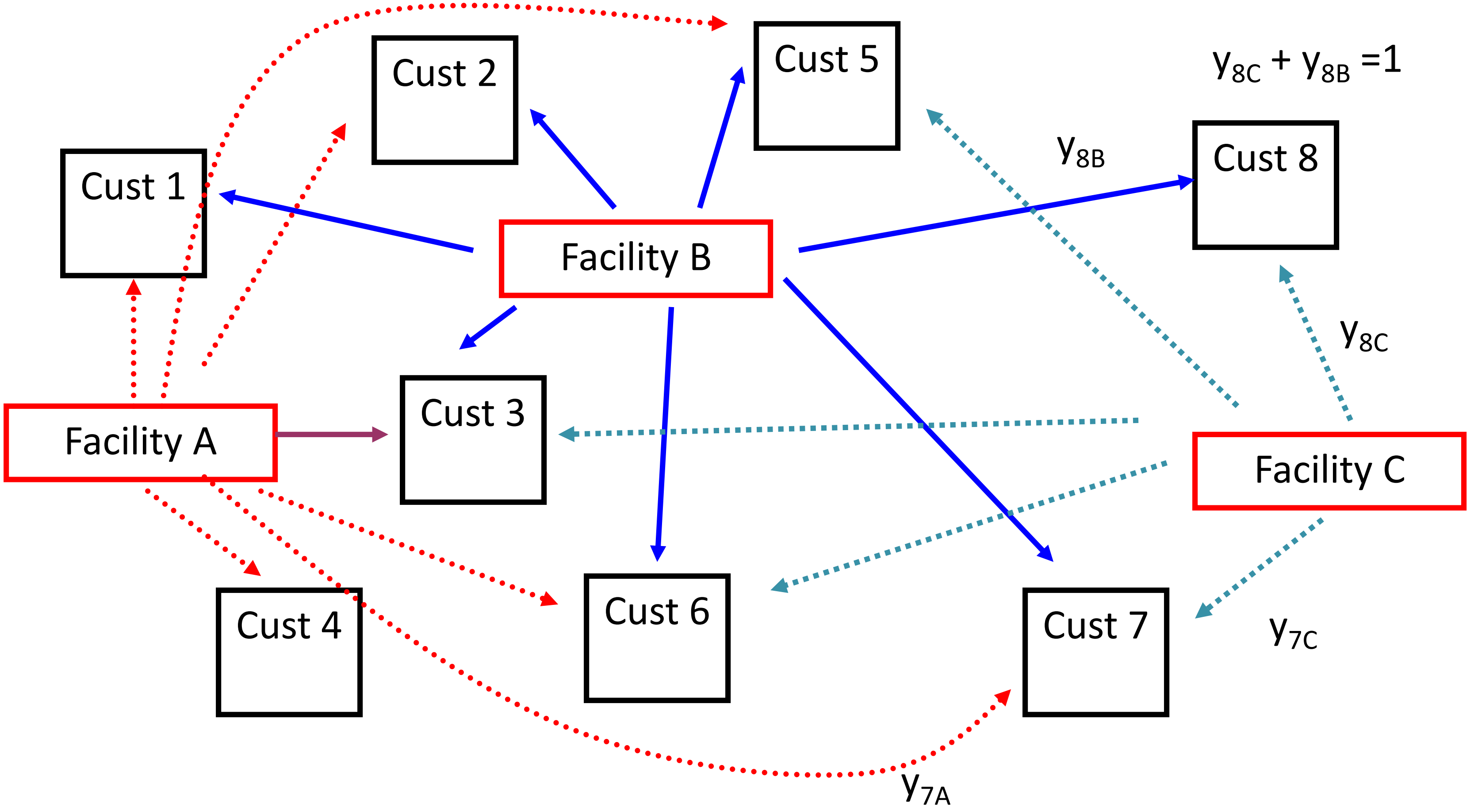
$$TC(n \geq 2) = 5{,}000(n) + \text{travel cost} \geq 10{,}000$$

  - When there is more than one coffee shop, the fixed cost per coffee shop and the traveling cost will be larger than $10,000 which is larger than $6,820 which is the total cost when single coffee shop is placed. Therefore there is no need to consider more than 1 coffee shop.

Plant Location Problem

- What we know:
  - possible locations for new facilities
- Involves determination of:
  - optimum number of new facilities
  - which customers (of existing facilities) should be served by each new facility
- The objective is to:
  - Minimize the total cost of supplying all demand to all the customers

Plant Location Problem



m = 8 and n = 3

Plant Location Problem

$$\min z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} y_{ij} + \sum_{j=1}^{n} f_i x_j$$

$$s.t.$$

$$\sum_{i=1}^{m} y_{ij} \le m x_j \qquad j = 1,\ldots,n$$

$$\sum_{j=1}^{n} y_{ij} = 1 \qquad i = 1,\ldots,m$$

$$y_{ij} \ge 0$$

$$x_j = \{0,1\}$$

- **where**

m is the number of customers

n is the number of plant sites

y

$\boldsymbol{y_{ij}}$ is the proportion of customer ***i*** demand supplied by a plant site ***j***

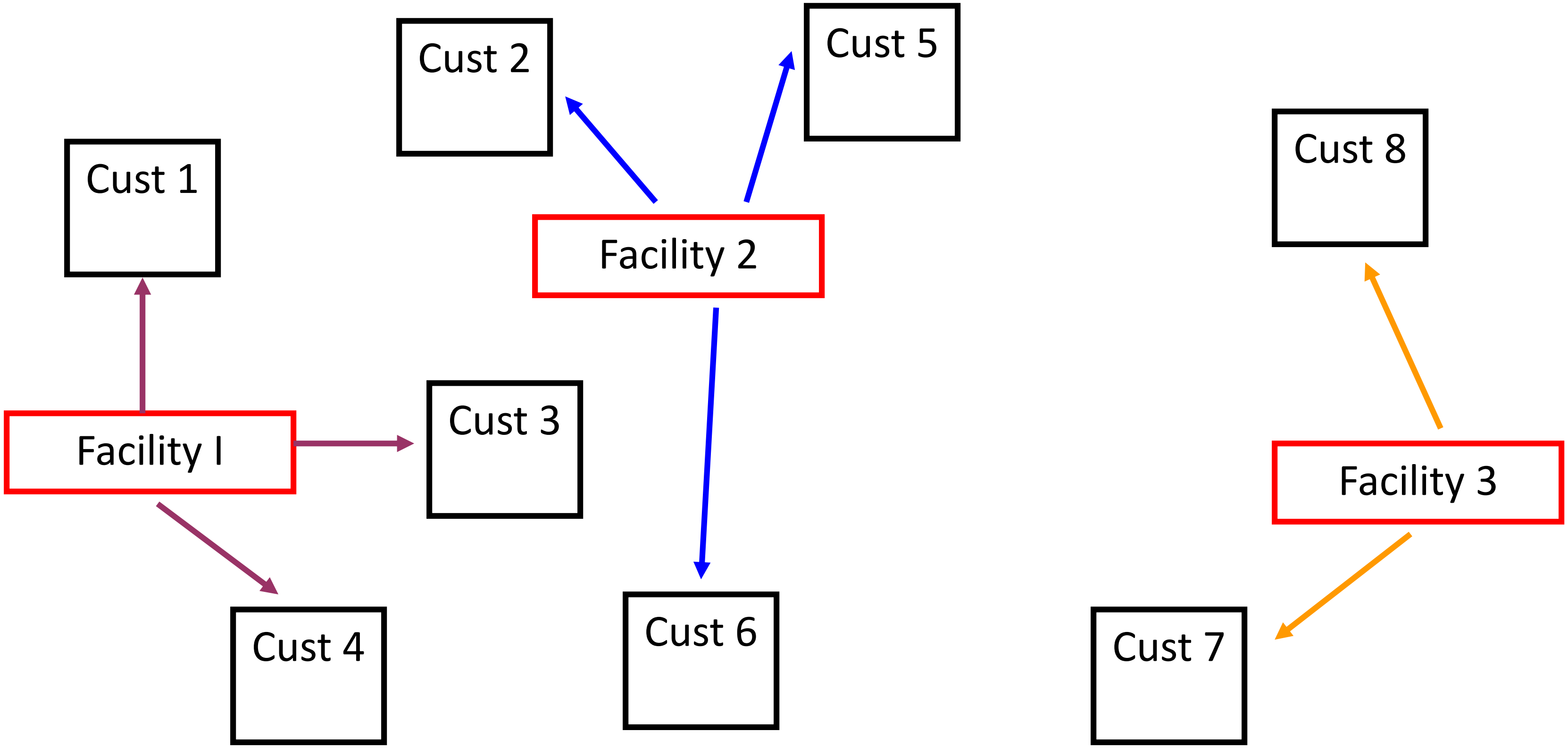
$\boldsymbol{x_j}$ is **1** if plant is located at ***j***, **0** otherwise

$\boldsymbol{c_{ij}}$ is the cost of supplying all demand of customer ***i*** from plant located at site ***j***

$\boldsymbol{f_j}$ is the fixed cost of locating the plant at site ***j***

Plant Location Problem

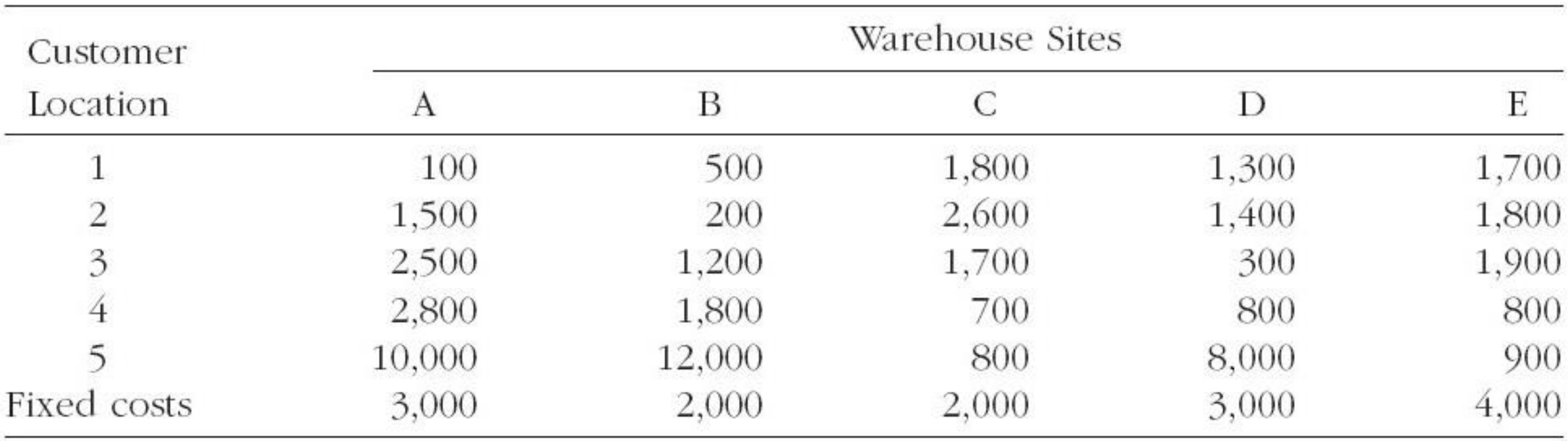
Simplified Model



Plant Location Problem

Example 1

- There are 5 existing customers (1,2,3,4 and 5).
- 5 sites are being considered for new warehouse location (A,B,C,D and E)
- **Each customer can be served by one warehouse only**
- The table below shows all the annual costs for each alternative site
- ➢ Which location should be selected if we want to build only one warehouse?
- ➢ If it was already decided that two warehouses are to be built - on site B and on site C, how many additional warehouses should be built and on which of the considered sites?



Customer	Warehouse Sites				
Location	A	B	C	D	E
1	100	500	1,800	1,300	1,700
2	1,500	200	2,600	1,400	1,800
3	2,500	1,200	1,700	300	1,900
4	2,800	1,800	700	800	800
5	10,000	12,000	800	8,000	900
Fixed costs	3,000	2,000	2,000	3,000	4,000

Plant Location Problem

Example 1

Customer Locations		Warehouse Locations				
		A	B	C	D	E
Cost of supplying the demand	1	100	500	1800	1300	1700
	2	1500	200	2600	1400	1800
	3	2500	1200	1700	300	1900
	4	2800	1800	700	800	800
	5	10000	12000	800	8000	900
Fixed Cost		3000	2000	2000	3000	4000
Total Annual Cost if selected		19,900	17,700	**9,600**	14,800	11,100

If only one warehouse is going to be built, location C should be selected.

Plant Location Problem

Example 1

- If two warehouses are built: one on site B and one on site C, then C will serve customers 4 and 5, and B will serve customers 1, 2 and 3:
- The total cost would be: TC=500+200+1200+2000+ 700+800+2000 = 7,400

Customer Locations		Warehouse Locations				
		A	**B**	**C**	D	E
Cost of supplying the demand	1	100	**500**	1800	1300	1700
	2	1500	**200**	2600	1400	1800
	3	2500	**1200**	1700	300	1900
	4	2800	1800	**700**	800	800
	5	10000	12000	**800**	8000	900
Fixed Cost		3000	**2000**	**2000**	3000	4000
Total Annual Cost if selected		0	**3,900**	**3,500**	0	0

Plant Location Problem

Example 1

- If we consider adding the third warehouse, we can calculate for each candidate site **Net Annual Savings (NAS)** :

- NAS **(A)** = 500 - 100 - 3000 = -2600

Customer Locations		Warehouse Locations				
		A	**B**	**C**	D	E
Cost of supplying the demand	1	**100**	**500**	1800	1300	1700
	2	1500	**200**	2600	1400	1800
	3	2500	**1200**	1700	300	1900
	4	2800	1800	**700**	800	800
	5	10000	12000	**800**	8000	900
Fixed Cost		**3000**	**2000**	**2000**	3000	4000
Total Annual Cost if selected		**3100**	**3900**	**3500**	0	0

Plant Location Problem

Example 1

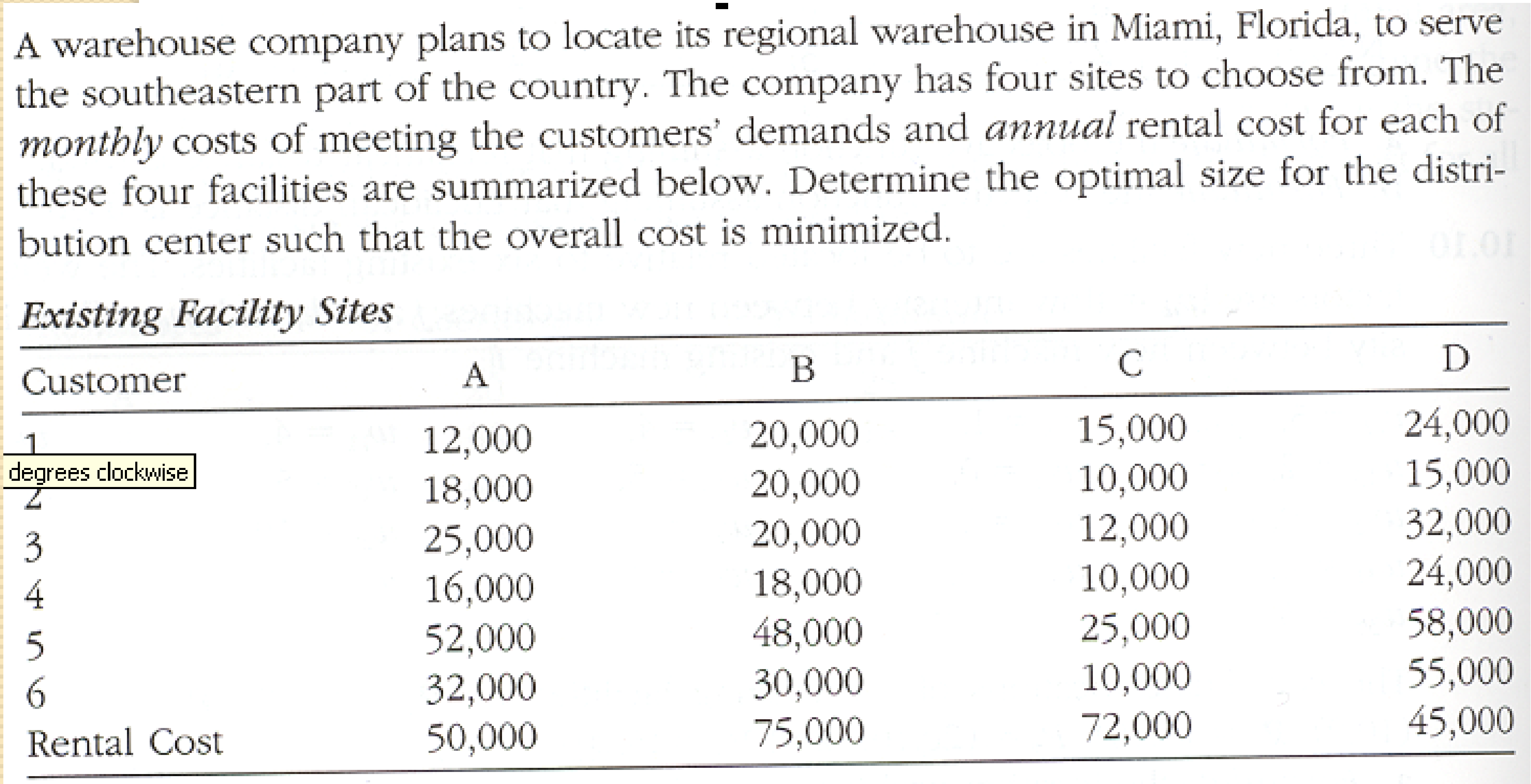
- If we consider adding the third warehouse, we can calculate for each candidate site **Net Annual Savings (NAS)** :
- NAS **(D)** = 1200 - 300 - 3000 = -2100

Customer Locations		Warehouse Locations				
		A	**B**	**C**	**D**	E
Cost of supplying the demand	1	100	**500**	1800	1300	1700
	2	1500	**200**	2600	1400	1800
	3	2500	**1200**	1700	**300**	1900
	4	2800	1800	**700**	800	800
	5	10000	12000	**800**	8000	900
Fixed Cost		3000	**2000**	**2000**	**3000**	4000
Total Annual Cost if selected		3100	**3900**	**3500**	**3300**	0

No additional warehouses are justified. Based on the circumstances, two warehouses placed at the sites B and C are the best solution.

Plant Location Problem

Example 2

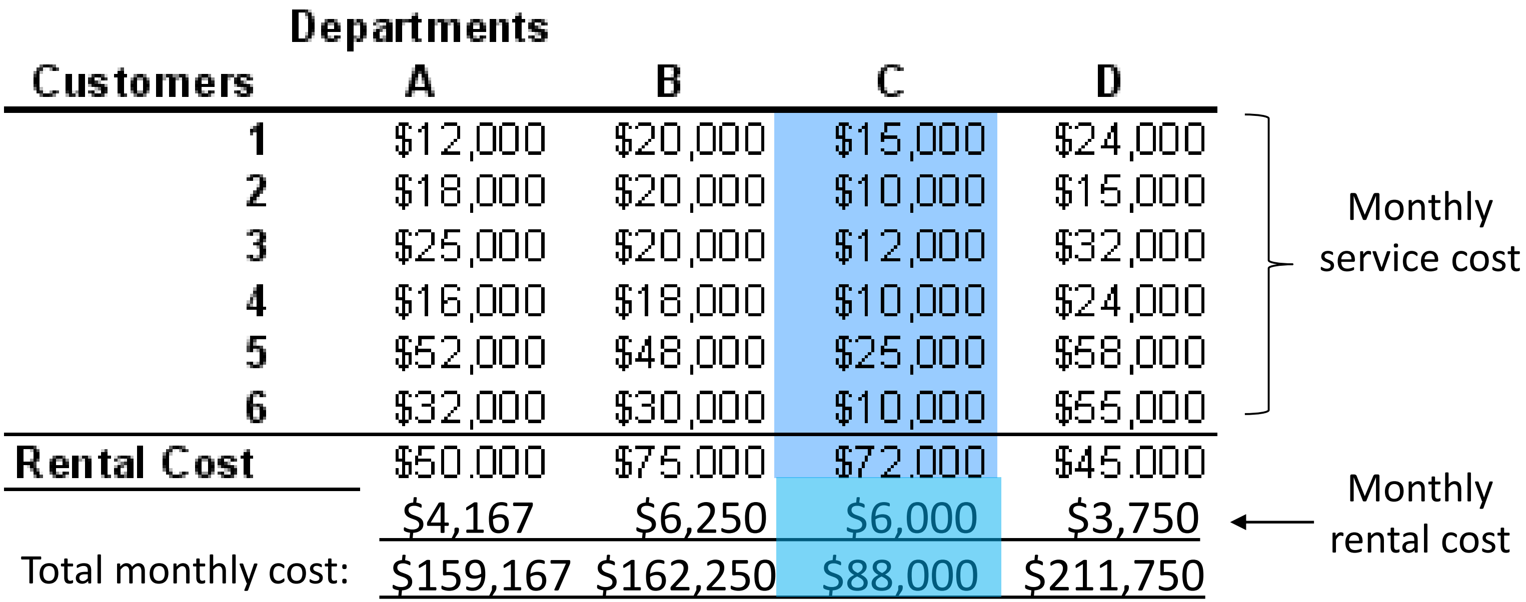


A warehouse company plans to locate its regional warehouse in Miami, Florida, to serve the southeastern part of the country. The company has four sites to choose from. The *monthly* costs of meeting the customers' demands and *annual* rental cost for each of these four facilities are summarized below. Determine the optimal size for the distribution center such that the overall cost is minimized.

Existing Facility Sites

Customer	A	B	C	D
1	12,000	20,000	15,000	24,000
2	18,000	20,000	10,000	15,000
3	25,000	20,000	12,000	32,000
4	16,000	18,000	10,000	24,000
5	52,000	48,000	25,000	58,000
6	32,000	30,000	10,000	55,000
Rental Cost	50,000	75,000	72,000	45,000

Plant Location Problem   Example 2



	Departments				
Customers	A	B	C	D	
1	$12,000	$20,000	$15,000	$24,000	Monthly service cost
2	$18,000	$20,000	$10,000	$15,000	
3	$25,000	$20,000	$12,000	$32,000	
4	$16,000	$18,000	$10,000	$24,000	
5	$52,000	$48,000	$25,000	$58,000	
6	$32,000	$30,000	$10,000	$55,000	
Rental Cost	$50.000	$75.000	$72.000	$45.000	
	$4,167	$6,250	$6,000	$3,750	Monthly rental cost
Total monthly cost:	$159,167	$162,250	$88,000	$211,750	

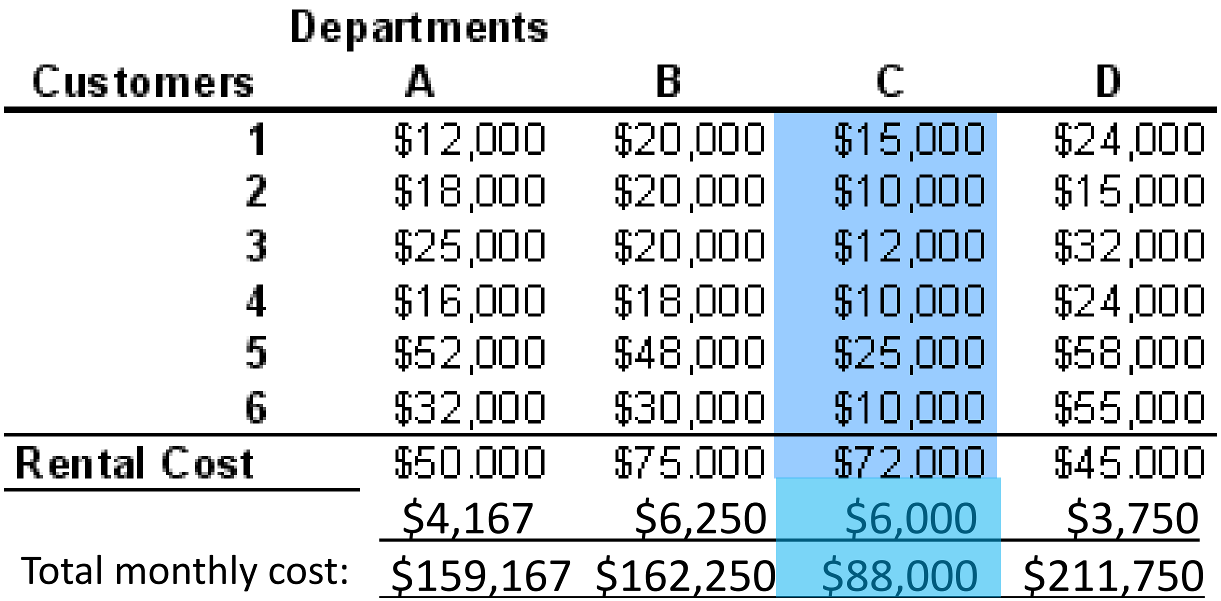
The monthly rental costs for facilities A, B, D, and D are $4,167, $6,250, $6,000, and $3,750, respectively.

a. 1 warehouse

Sites	A	B	C	D
Total cost per month	159,167	162,250	88,000	212,750

Select site C, TC = 88,000/month = 1,045,000/yr.

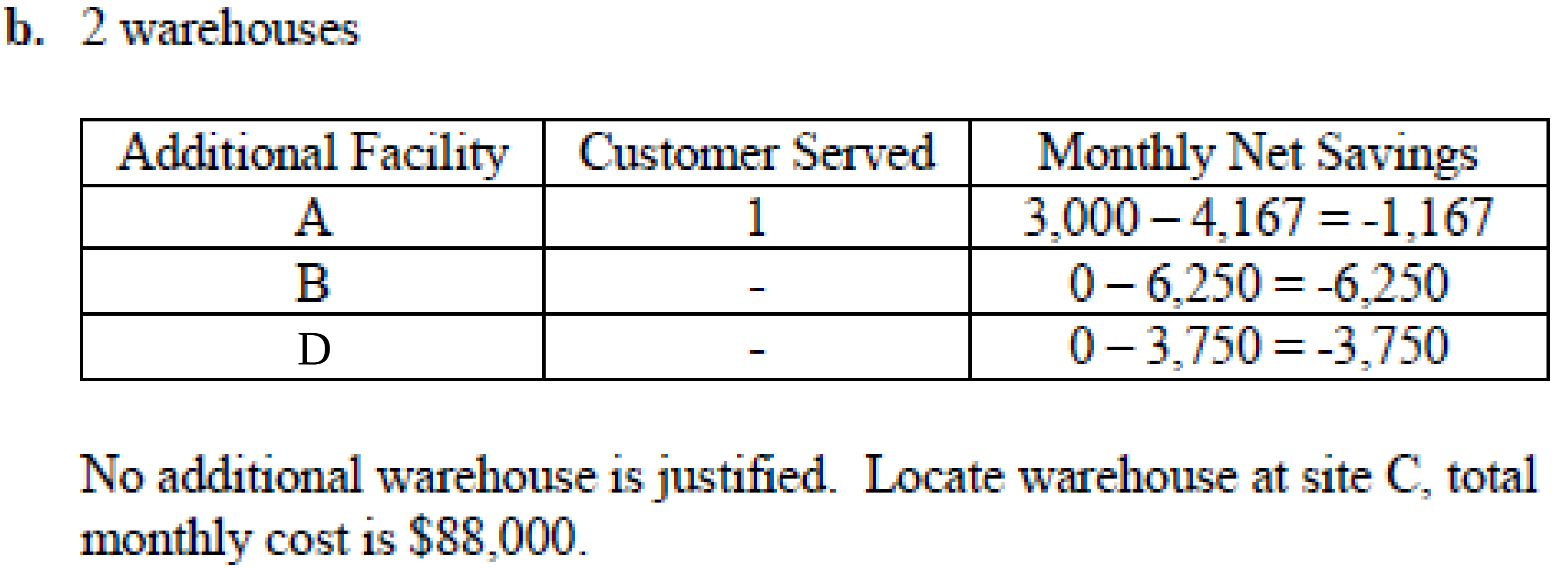
Plant Location Problem   Example 2



	Departments			
Customers	A	B	C	D
1	$12,000	$20,000	$15,000	$24,000
2	$18,000	$20,000	$10,000	$15,000
3	$25,000	$20,000	$12,000	$32,000
4	$16,000	$18,000	$10,000	$24,000
5	$52,000	$48,000	$25,000	$58,000
6	$32,000	$30,000	$10,000	$55,000
Rental Cost	$50,000	$75,000	$72,000	$45,000
	$4,167	$6,250	$6,000	$3,750
Total monthly cost:	$159,167	$162,250	$88,000	$211,750

Monthly service cost

Monthly rental cost



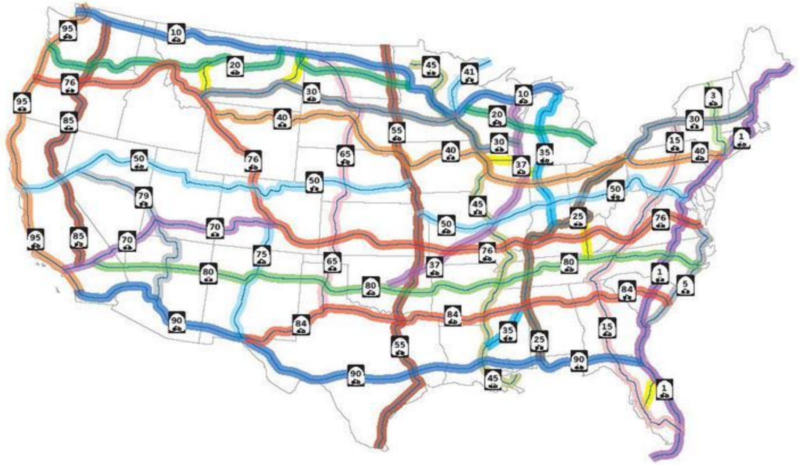
b. 2 warehouses

Additional Facility	Customer Served	Monthly Net Savings
A	1	3,000 – 4,167 = -1,167
B	-	0 – 6,250 = -6,250
D	-	0 – 3,750 = -3,750

No additional warehouse is justified. Locate warehouse at site C, total monthly cost is $88,000.

Network Location Problems

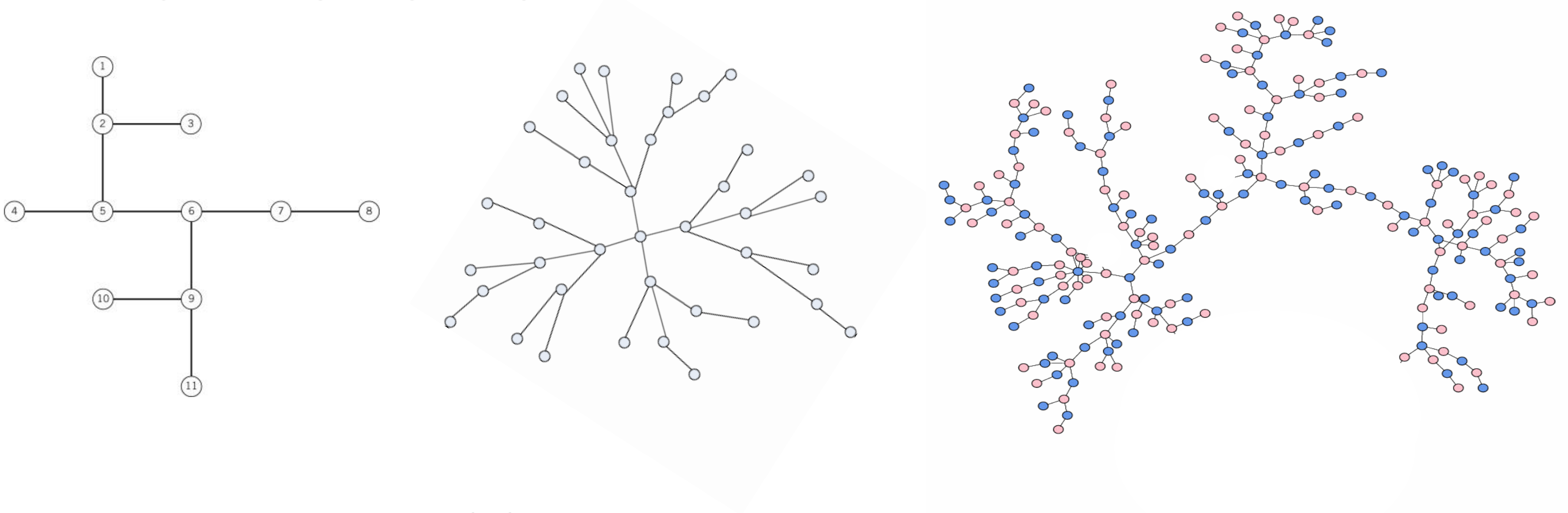
- One or more new facilities are located on a network
- The network represents the actual distances between new and existing facilities
- The travel path is used to calculate the distance
    - More difficult to solve because of multiple paths connecting any two points on the network - **cyclical networks**



- We will consider only **tree networks**

Network Location Problems

- **Tree network** does not have cycles
  - A unique path exists between any two points on the network



- Median problems (minisum equivalent)
  - n-Median problem
  - 1-Median problem
- Center problem (minimax equivalent)
  - n-Center problem
  - 1-Center problem

1-Median Problem

- Objective is to minimize the sum of weighted distances between the new facility and all the existing ones

$$Minimize\ f(x) = \sum_{i=1}^{m} w_i\, d(x,\ v_i)$$

$d(x,\ v_i)$ ... distance between a point on the tree (*x*) and vertex *i*

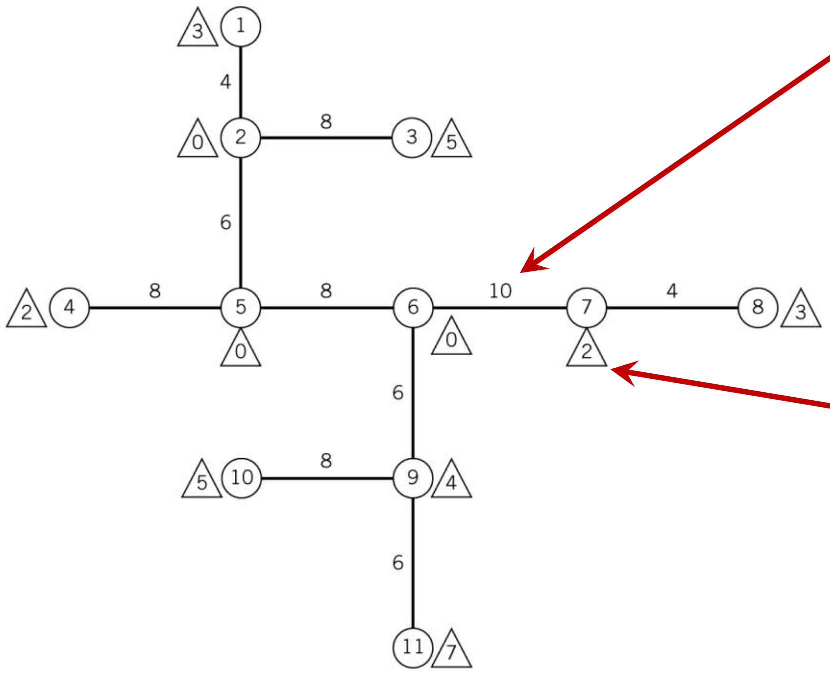
  - However, distances are not considered in this problem and only spatial relative positions are important

- **Chinese algorithm**
  - Procedure:
    - Trim a branch from the tree that has the smallest weight and add the weight to the vertex from which the branch emanated
    - Break ties arbitrarily
    - Continue the process until only one vertex remains (= location of the new facility)

1-Median Problem – Chinese Algorithm

Example

- A central warehouse is to be located close to 11 existing manufacturing facilities which it will serve. The road network among the facilities is simplified by the tree network shown below. Due to terrain difficulties, it is impossible to travel from *v5* to v*10* without passing through *v6* and *v9*. Similarly, to travel from *v3* to *v6* one needs to go through *v2* and *v5*. Distances and the travel frequencies are shown below. Find the location for the new warehouse through Chinese algorithm.



Distance:

- The distance between 2 adjacent nodes

Weight:

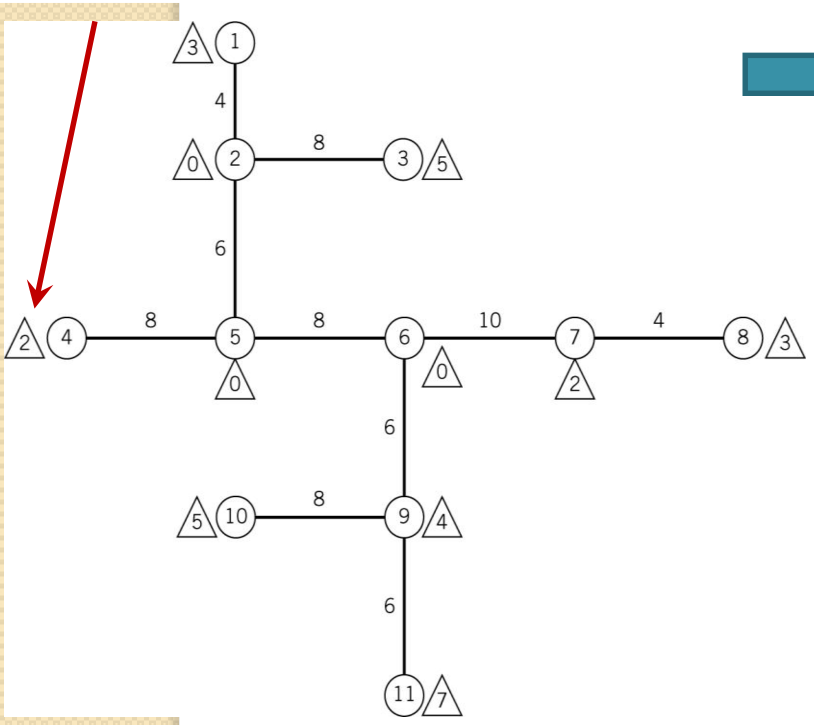
- The number of travel times between the locations of existing facilities and the warehouse
- It does not include "pass-through travel"

1-Median Problem – Chinese Algorithm

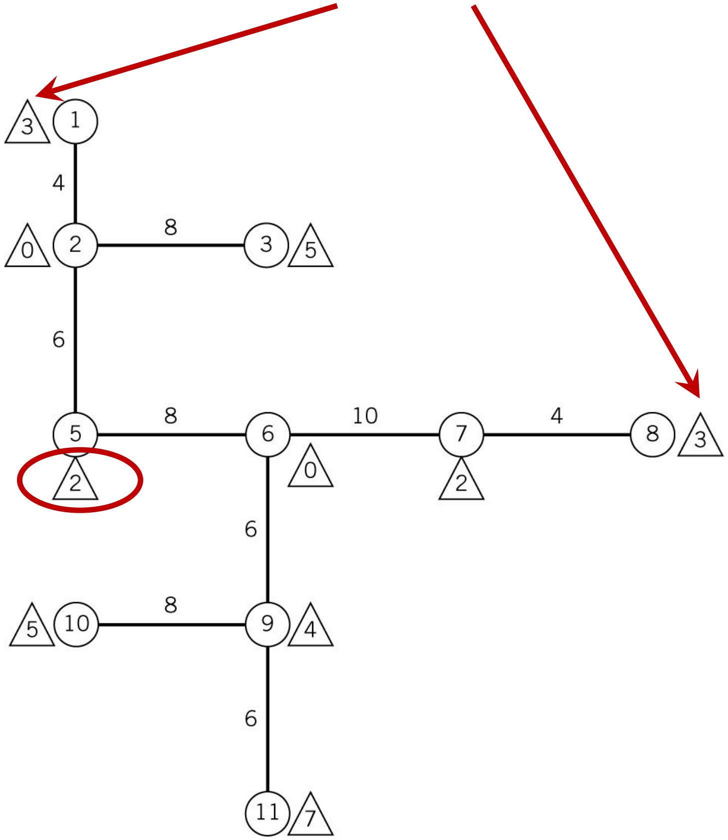
Example

- Trim a branch from the tree that has the smallest weight and add the weight to the vertex from which the branch emanated
- Trim node 4
- Trim node 1

The smallest weight



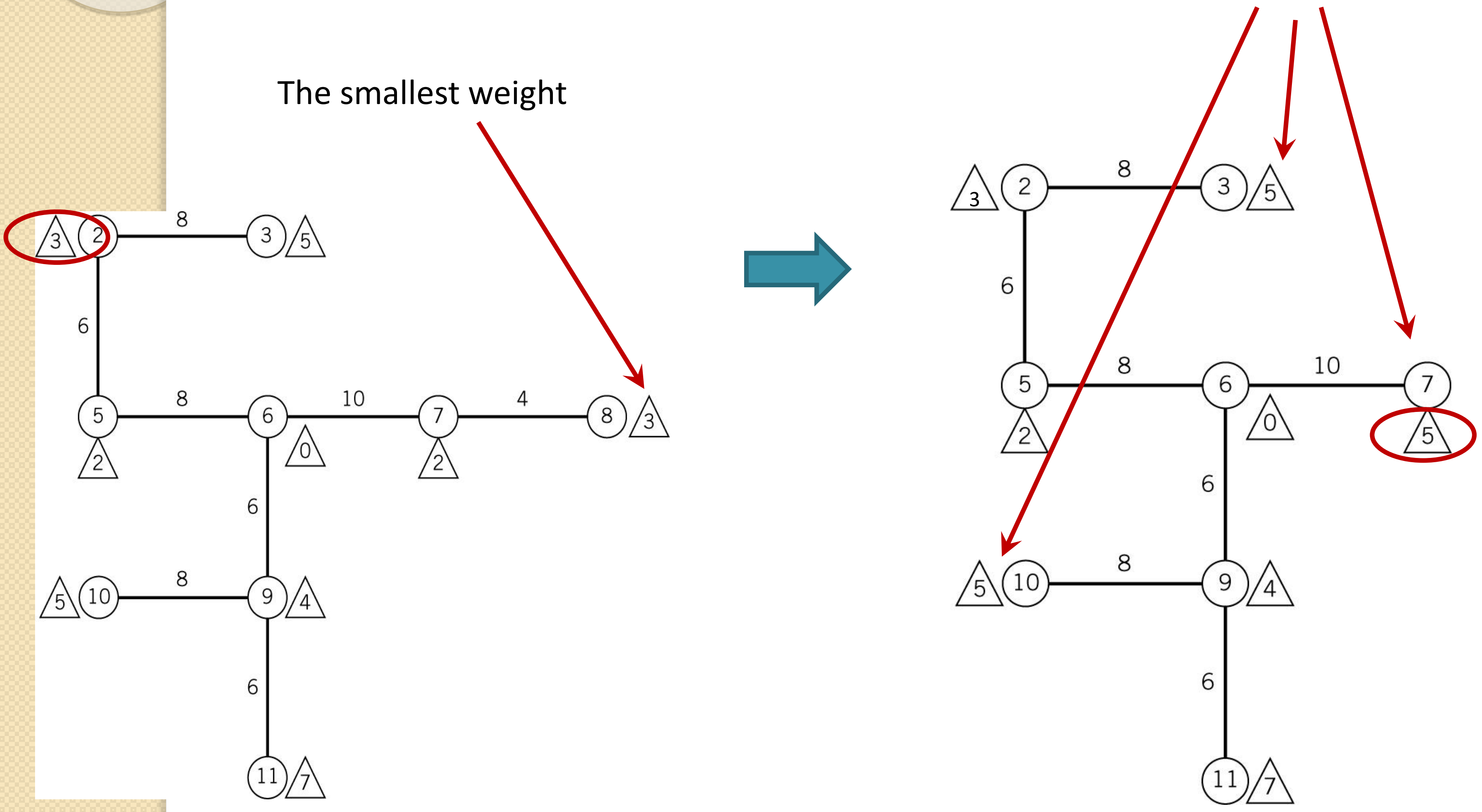
The smallest weight
(Node 1 arbitrarily selected)



1-Median Problem – Chinese Algorithm

Example

- Trim node 8
- Trim node 3



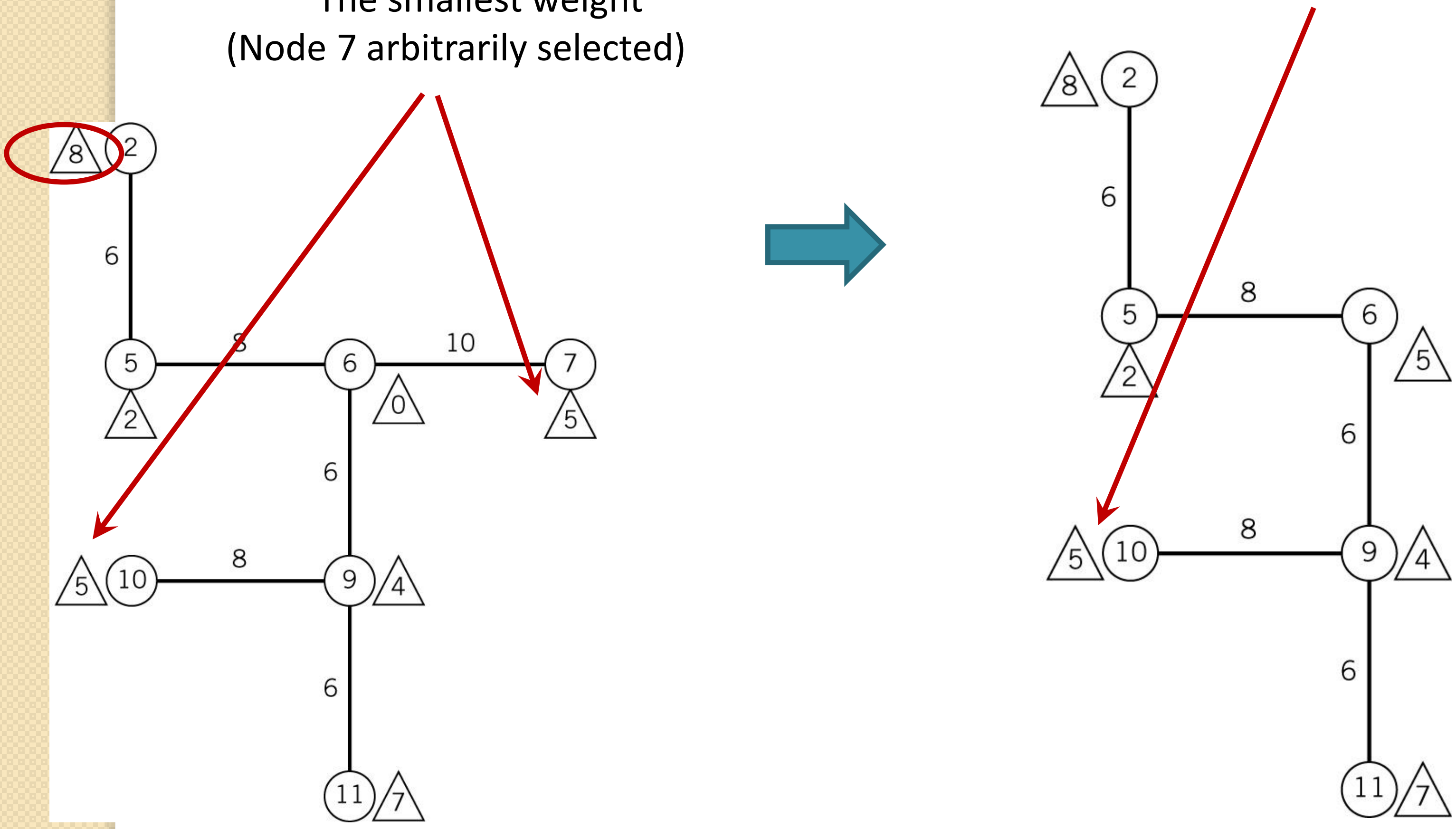
The smallest weight

The smallest weight
(Node 3 arbitrarily selected)

1-Median Problem – Chinese Algorithm

Example

- Trim node 7
- Trim node 10



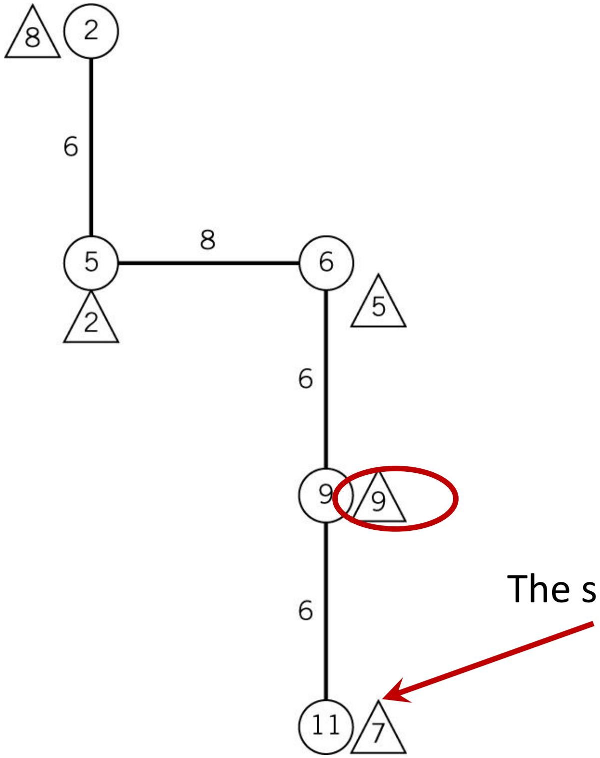
The smallest weight
(Node 7 arbitrarily selected)

The smallest weight

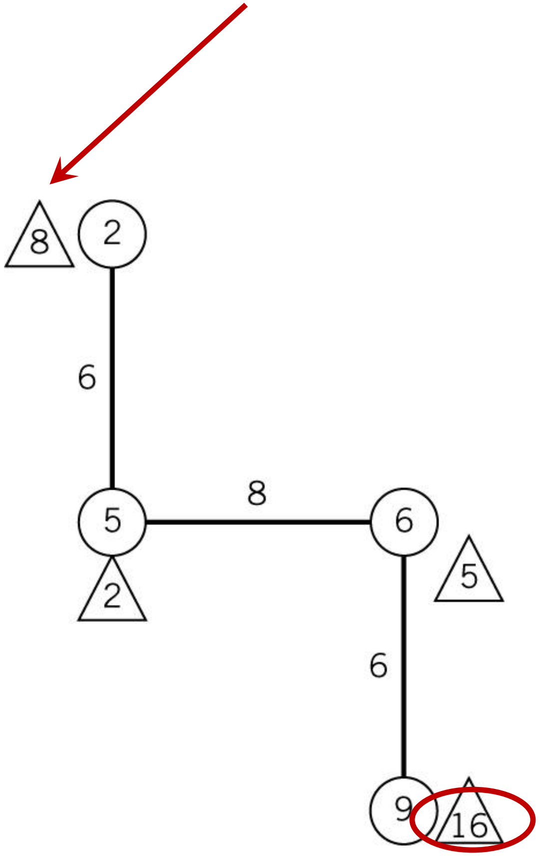
1-Median Problem – Chinese Algorithm

Example

- Trim node 11
- Trim node 2



The smallest weight

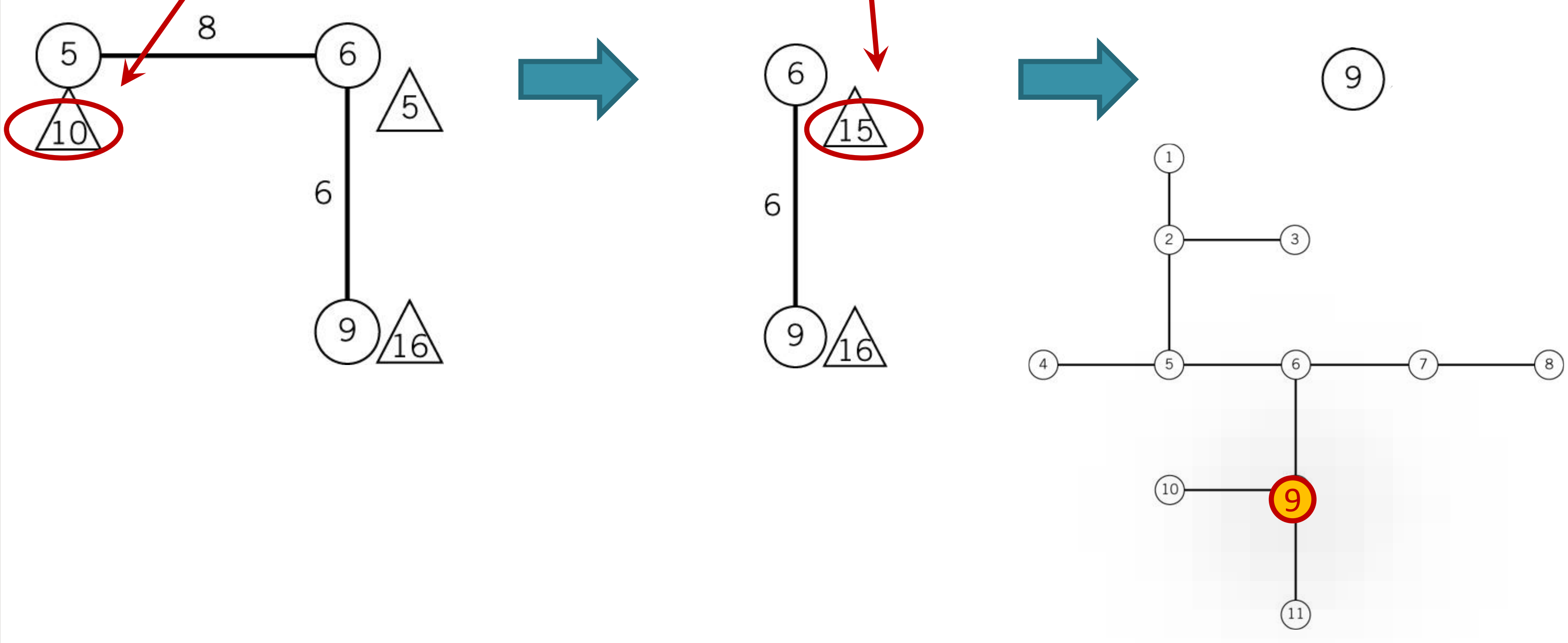


The smallest weight

1-Median Problem – Chinese Algorithm

Example

- Trim node 5
- Trim node 6



The smallest weight

The smallest weight

Node 9 is an
optimum
location for the
warehouse
$x = v9$

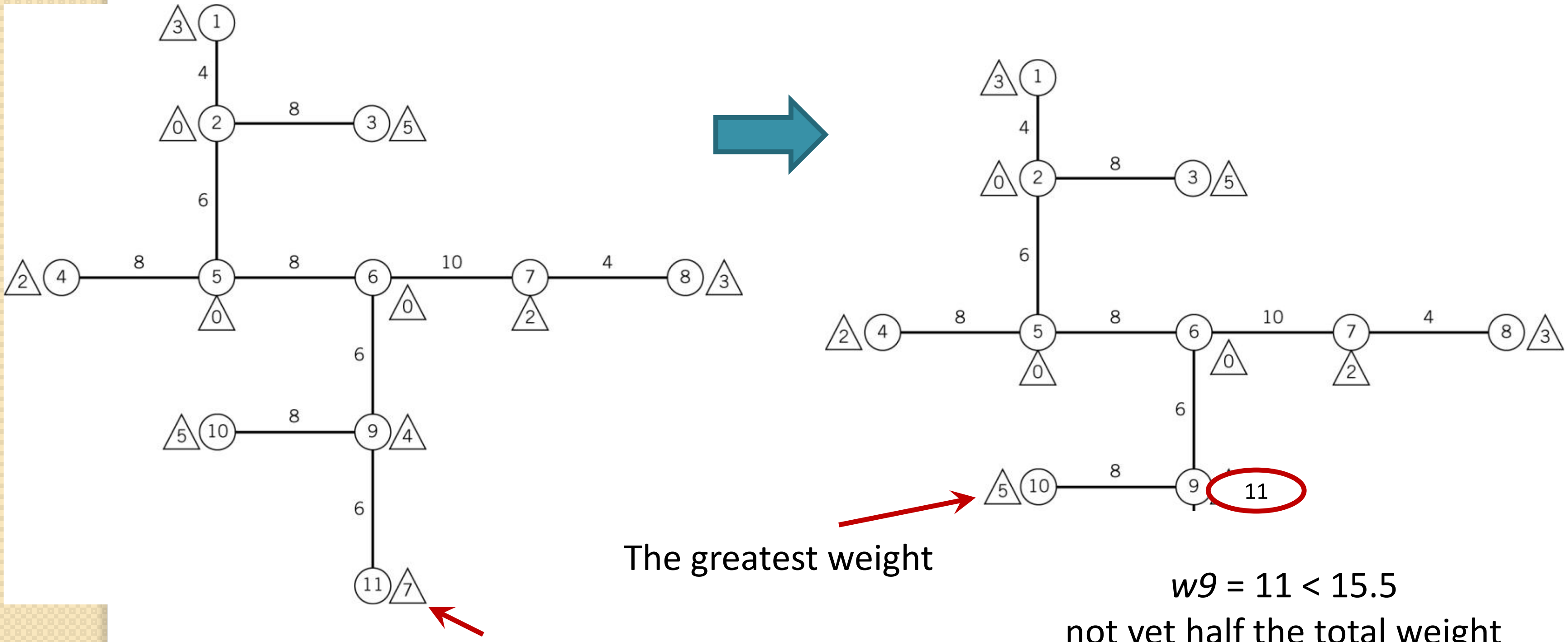
1-Median Problem – Majority Algorithm

- **Majority algorithm** finds an optimum location which has half the total weight to either side of it
- Procedure
    - Calculate half the total weight on the tree
    - Trim a branch from the tree that has the greatest weight and add the weight to the vertex from which the branch emanated
    - Break ties arbitrarily
    - Continue the process until at least half the total weight is at one node (= location of the new facility)

1-Median Problem – Majority Algorithm

Example – the same problem

- Calculate half the total weight on the tree
    - Half the total weight = (3+0+5+2+0+0+2+3+5+4+7)/2=31/2=**<u>15.5</u>**
- Trim a branch from the tree that has the <u>greatest</u> weight and add the weight to the vertex from which the branch emanated → *Trim node 11*
- See if at least half the total weight is at one node → *Not yet, trim node 10*



The greatest weight

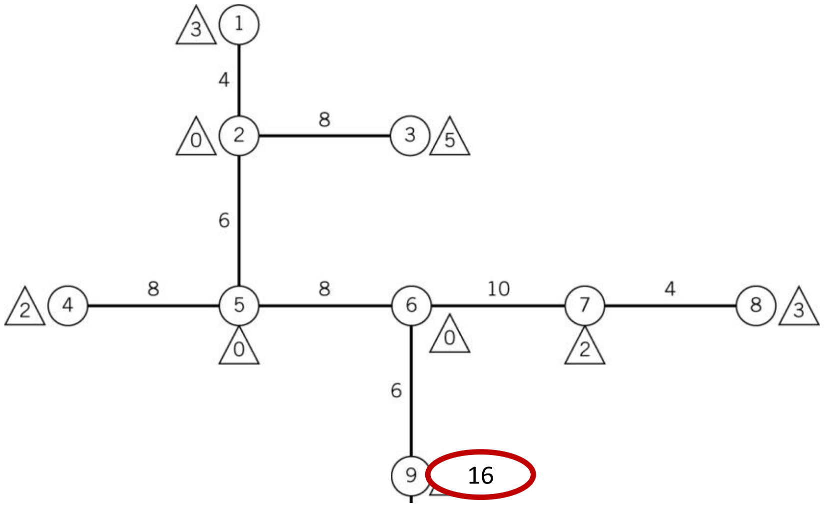
The greatest weight

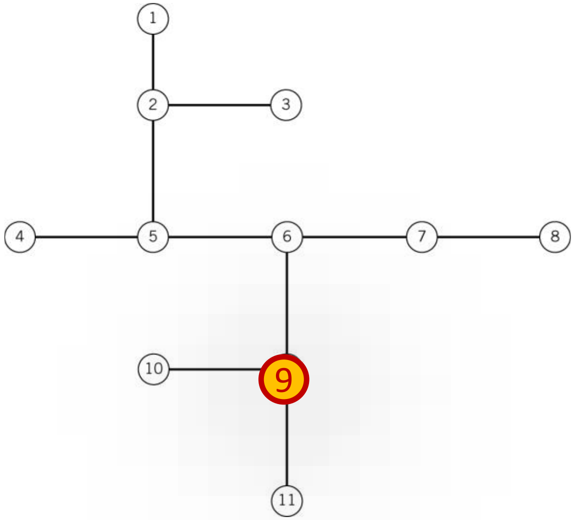
w9 = 11 < 15.5
not yet half the total weight

1-Median Problem – Majority Algorithm

Example – the same problem

- Trimming node 10 and adding its weight to node 9 resulted in half the total weight to be in node 9 → optimum location





w9 = 16 > 15.5
half the total weight is at *v9*

v9 is the optimum location for the warehouse

1-Center Problem

- Objective is to minimize the maximum weighted distance between a new facility and any other existing facility

$$Minimize\, f(x) = \max_{1 \le i \le m} \{w_1 d(x, v_1), \ldots, w_m d(x, v_m)\}$$

- The new facility is located at a point x* in the tree network. To find the point we need to:
  - Calculate **b_{ij} values** for all the pairs of nodes

$$b_{ij} = w_i w_j d(v_i, v_j)/(w_i + w_j)$$

  - Determine the **maximum value b_{st}**

$$b_{st} = \max\{b_{ij}: 1 = i < j = m\}$$

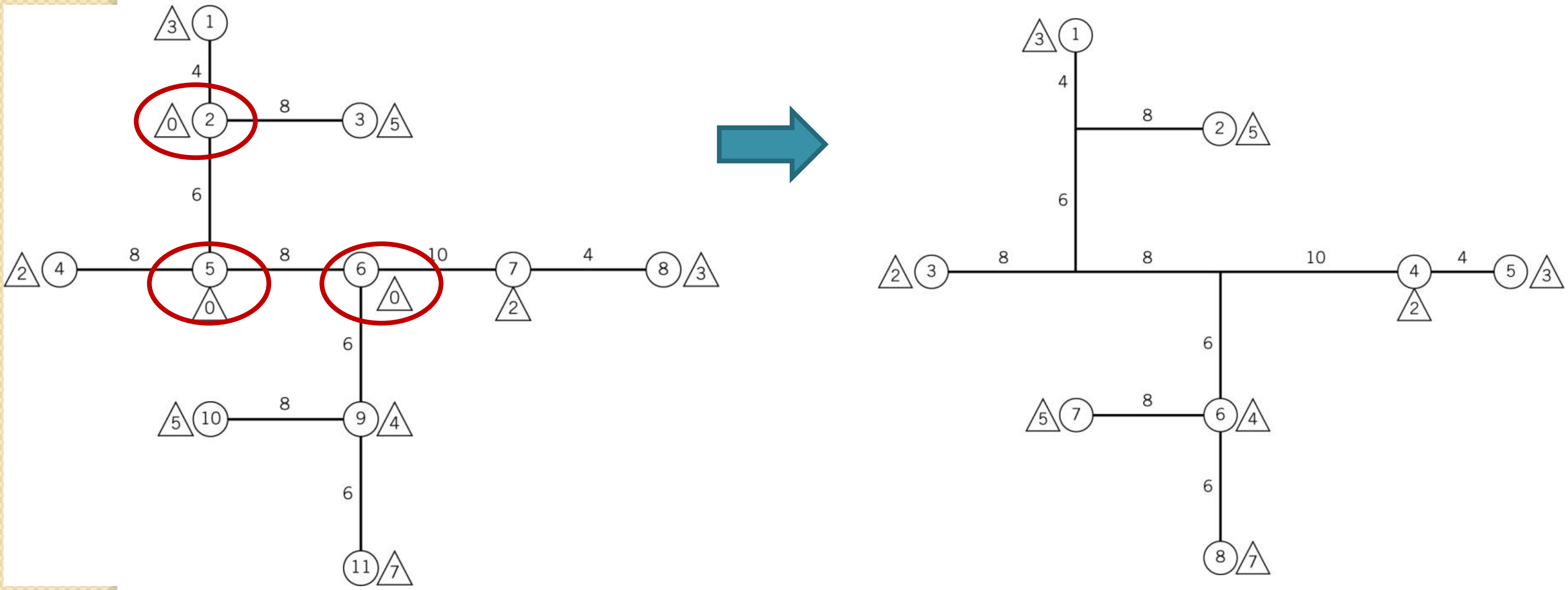
  - **x*** is located on the path connecting v_s and v_t

1-Center Problem

Example – the same problem

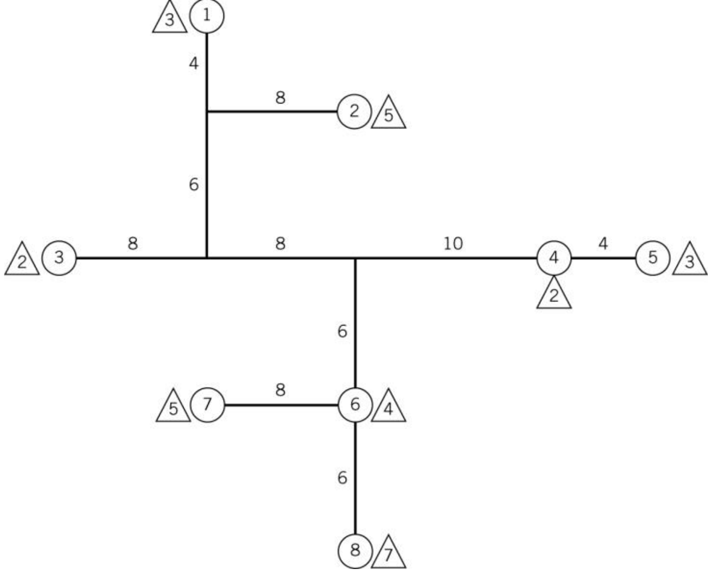
- Consider the previous problem with the warehouse
  - Only vertices with *positive-valued weights* are considered.

    → Three vertices are removed from the tree and the remaining ones are renumbered



11 node network → 8 node network

1-Center Problem - **Example**



- The new facility is located at a point x* in the tree network. To find the point we need to:
  - Calculate b_{ij} **values** for all the pairs of nodes

$$b_{ij} = w_i w_j d(v_i, v_j)/(w_i + w_j)$$

b_{12} = 3*5*(4+8)/(3+5) = **22.5**

b_{36} = 2*4*(8+8+6)/(2+4) = **29.333**

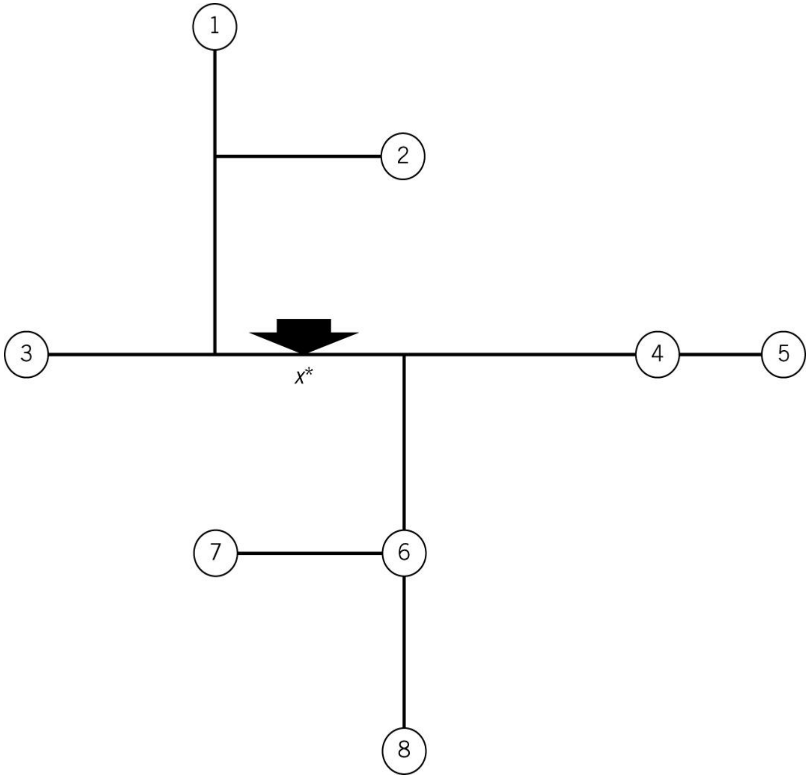
	Machine *j*							
Machine *i*	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	0	22.5	21.6	33.6	48	41.143	60	63
2		0	31.429	45.7144	67.5	62.222	90	99.167
3			0	22	31.2	29.333	42.857	43.556
4				0	4.8	21.333	34.286	34.222
5					0	34.386	52.5	54.6
6						0	17.778	15.273
7							0	40.833
8								0

max

1-Center Problem

Example

- Determine the **maximum value b_{st}**
  - $b_{28} = 99.167$ corresponds to vertices 2 and 8.
- **x*** is located on the path connecting vertex 2 and vertex 8



Next lecture

- Warehouse operations